Example 10.18: Of those women who are diagnosed to have early-stage breast cancer, one-third eventually die of the disease. Suppose an NGO launch a screening programme to provide for the early detection of breast cancer and to increase the survival rate of those diagnosed to have the disease. A random sample of 200 women was selected from among those who were periodically screened and who were diagnosed to have the disease. If 164 women in the sample of 200 survive the disease, can screening programme be considered effective? Test using $\alpha = 0.01$ level of significance and explain the conclusions from your test.

Solution: Let us take the null hypothesis that the screening programme was effective, that is,

$$H_0: p = 1 - (1/3) = 2/3 \text{ and } H_1: p > 2/3$$

Given $n = 200, p = 2/3, q = 1/3$ and $\alpha = 0.05$. Applying the z-test statistic,

$$z = \frac{x - np}{\sqrt{npq}} = \frac{164 - 200 \times (2/3)}{\sqrt{200 \times (2/3)(1/3)}} = \frac{164 - 133.34}{\sqrt{44.45}}$$

$$= \frac{30.66}{6.66} = 4.60$$

Since $z_{cal} = 4.60$ is greater than its critical value $z_\alpha = 2.33$ at $\alpha = 0.01$ significance level, the null hypothesis, H_0 is rejected. Hence we conclude that the screening programme was not effective.

Self-Practice Problems 10B

10.11 A company manufacturing a certain type of breakfast cereal claims that 60 per cent of all housewives prefer that type to any other. A random sample of 300 housewives contains 165 who do prefer that type. At 5 per cent level of significance, test the claim of the company.

10.12 An auditor claims that 10 per cent of a company's invoices are incorrect. To test this claim a random sample of 200 invoices is checked and 24 are found to be incorrect. At 1 per cent significance level, test whether the auditor's claim is supported by the sample evidence.

10.13 A sales clerk in the department store claims that 60 per cent of the shoppers entering the store leave without making a purchase. A random sample of 50 shoppers showed that 35 of them left without buying anything. Are these sample results consistent with the claim of the sales clerk? Use a significance level of 0.05. [*Delhi Univ., MBA, 1998, 2001*]

10.14 A dice is thrown 49,152 times and of these 25,145 yielded either 4, 5, or 6. Is this consistent with the hypothesis that the dice must be unbiased?

10.15 A coin is tossed 100 times under identical conditions independently yielding 30 heads and 70 tails. Test at 1 per cent level of significance whether or not the coin is unbiased. State clearly the null hypothesis and the alternative hypothesis.

10.16 Before an increase in excise duty on tea, 400 people out of a sample of 500 persons were found to be tea drinkers. After an increase in the duty, 400 persons were known to be tea drinkers in a sample of 600 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty? [*Delhi Univ., MCom, 1998; MBA, 2000*]

10.17 In a random sample of 1000 persons from UP 510 were found to be consumers of cigarettes. In another sample of 800 persons from Rajasthan, 480 were found to be consumers of cigarettes. Do the data reveal a significant difference between UP and Rajasthan so far as the proportion of consumers of cigarettes in concerned? [*MC Univ., M.Com, 1996*]

10.18 In a random sample of 500 persons belonging to urban areas, 200 are found to be using public transport. In another sample of 400 persons belonging to rural area 200 area found to be using public transport. Do the data reveal a significant difference between urban and rural areas so far as the proportion of commuters of public transport is concerned (use 1 per cent level of significance). [*Bharathidasan Univ., MCom, 1998*]

10.19 A machine puts out 10 defective units in a sample of 200 units. After the machine is overhauled it puts out 4 defective units in a sample of 100 units. Has the machine been improved? [*Madras Univ., MCom, 1996*]

10.20 500 units from a factory are inspected and 12 are found to be defective, 800 units from another factory are inspected and 12 are found to be defective. Can it be concluded that at 5 per cent level of significance production at the second factory is better than in first factory? [*Kurukshetra Univ., MBA, 1996; Delhi Univ., MBA, 2002*]

10.21 In a hospital 480 female and 520 male babies were born in a week. Do these figures confirm the hypothesis that females and males are born in equal number? [*Madras Univ., MCom, 1997*]

10.22 A wholesaler of eggs claims that only 4 per cent of the eggs supplied by him are defective. A random sample of 600 eggs contained 36 defectives. Test the claim of the wholesaler. [*IGNOU, 1997*]

Hints and Answers

10.11 Let $H_0 : p = 60$ per cent and $H_1 : p < 60$ per cent (One tailed test)

Given, sample proportion, $\bar{p} = 165/300 = 0.55$; $n = 300$ and $z_\alpha = 1.645$ at $\alpha = 5$ per cent

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{pq}{n}}} = \frac{0.55 - 0.60}{\sqrt{\frac{0.60 \times 0.40}{300}}} = -1.77$$

Since z_{cal} (= – 1.77) is less than its critical value $z_\alpha = -1.645$, the H_0 is rejected. Percentage preferring the breakfast cereal is lower than 60 per cent.

10.12 Let $H_0 : p = 10$ per cent and $H_1 : p \neq 10$ per cent (Two-tailed test)

Given, sample proportion, $\bar{p} = 24/200 = 0.12$; $n = 200$ and $z_{\alpha/2} = 2.58$ at $\alpha = 1$ per cent

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{pq}{n}}} = \frac{0.12 - 0.10}{\sqrt{\frac{0.10 \times 0.90}{200}}} = 0.943$$

Since z_{cal} (= 0.943) is less than its critical value $z_{\alpha/2} = 2.58$, the H_0 is accepted. Thus the percentage of incorrect invoices is consistent with the auditor's claim of 10 per cent.

10.13 Let $H_0 : p = 60$ per cent and $H_1 : p \neq 60$ per cent (Two-tailed test)

Given, sample proportion, $\bar{p} = 35/60 = 0.70$; $n = 50$ and $z_{\alpha/2} = 1.96$ at $\alpha = 5$ per cent

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{pq}{n}}} = \frac{0.70 - 0.60}{\sqrt{\frac{0.60 \times 0.40}{50}}} = 1.44$$

Since z_{cal} (= 1.44) is less than its critical value $z_{\alpha/2}$ = 1.96, the H_0 is accepted. Claim of the sales clerk is valid.

10.14 Let $H_0 : p = 50$ per cent and $H_1 : p \neq 50$ per cent (Two-tailed test)

Given, sample proportion of success p = 25,145/ 49,152 = 0.512; n = 49,152

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{pq}{n}}} = \frac{0.512 - 0.50}{\sqrt{\frac{0.50 \times 0.50}{49{,}152}}} = \frac{0.012}{0.002} = 6.0$$

Since z_{cal} (= 6.0) is more than its critical value $z_{\alpha/2}$ = 2.58 at a = 0.01, the H_0 is rejected, Dice is biased.

10.15 Let $H_0 : p = 50$ per cent and $H_1 : p \neq 50$ per cent (Two-tailed test)

Given, $n = 100$, sample proportion of success $\bar{p} = 30/100 = 0.30$ and $z_{\alpha/2} = 2.58$ at $\alpha = 0.01$

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{pq}{n}}} = \frac{0.30 - 0.50}{\sqrt{\frac{0.50 \times 0.50}{100}}} = -\frac{0.20}{0.05} = -4$$

Since z_{cal} (= – 4) is less than its critical value $z_{\alpha/2}$ = – 2.58, the H_0 is rejected.

10.16 Let $H_0 : p = 400/500 = 0.80$ and $H_1 : p < 0.80$ (One-tailed test)

Given $n_1 = 500$, $n_2 = 600$, $\bar{p}_1 = 400/500 = 0.80$, $\bar{p}_2 = 400/600 = 0.667$ and $z_\alpha = 2.33$ at $\alpha = 0.01$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{400 + 400}{500 + 600} = 0.727;$$

$$q = 1 - 0.727 = 0.273$$

$$s_{\bar{p}_1 - \bar{p}_2} = \sqrt{\bar{p}\,\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{0.727 \times 0.273\left(\frac{1}{500} + \frac{1}{600}\right)} = 0.027$$

$$z = \frac{\bar{p}_1 - \bar{p}_2}{s_{\bar{p}_1 - \bar{p}_2}} = \frac{0.80 - 0.667}{0.027} = 4.93$$

Since z_{cal} (= 4.93) is more than its critical value z_α = 2.33, the H_0 is rejected. Decrease in the consumption of tea after the increase in the excise duty is significant.

10.17 Let $H_0 : p_1 = p_2$ and $H_1 : p_1 \neq p_2$ (Two-tailed test)

Given,UP: $n_1 = 1000$, $\bar{p}_1 = 510/1000 = 0.51$;
Rajasthan: $n_2 = 800$, $\bar{p}_2 = 480/800 = 0.60$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{510 + 480}{1000 + 800} = 0.55;$$

$$q = 1 - 0.55 = 0.45$$

$$s_{\bar{p}_1 - \bar{p}_2} = \sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{0.55 \times 0.45\left(\frac{1}{1000} + \frac{1}{800}\right)} = 0.024.$$

$$z = \frac{\bar{p}_1 - \bar{p}_2}{s_{\bar{p}_1 - \bar{p}_2}} = \frac{0.51 - 0.60}{0.024} = -3.75$$

Since z_{cal} (= – 3.75) is less than its critical value $z_{\alpha/2}$ = – 2.58, the H_0 is rejected. The proportion of consumers of cigarettes in the two states is significant.

10.18 Let $H_0 : p_1 = p_2$ and $H_1 : p_1 \neq p_2$ (Two-tailed test)

Given, Urban area: $n_1 = 500$, $\bar{p}_1 = 200/500 = 0.40$;
Rural area: $n_2 = 200$, $\bar{p}_2 = 200/400 = 0.50$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{200 + 200}{500 + 400} = 0.44;$$

$$q = 1 - p = 0.55$$

$$s_{\bar{p}_1 - \bar{p}_2} = \sqrt{\bar{p}\,\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{0.44 \times 0.55\left(\frac{1}{500} + \frac{1}{400}\right)} = 0.033$$

$$z = \frac{\bar{p}_1 - \bar{p}_2}{s_{\bar{p}_1 - \bar{p}_2}} = \frac{0.40 - 0.50}{0.033} = -3.03$$

Since z_{cal} = – 3.03 is less than its critical value $z_{\alpha/2}$ = – 2.58, the H_0 is rejected. Proportion of commuters of public transport in urban and rural areas is significant.

10.19 Let $H_0 : p_1 \le p_2$ and $H_1 : p_1 > p_2$ (One-tailed test)

Given, Before overhaul: $n_1 = 200$, $\bar{p}_1 = 10/200 = 0.05$;

After overhaul: $n_2 = 100$, $\bar{p}_2 = 4/100 = 0.04$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{10+4}{200+100} = 0.047;$$

$$q = 1 - p = 0.953$$

$$s_{\bar{p}_1-\bar{p}_2} = \sqrt{0.047 \times 0.953\left(\frac{1}{200} + \frac{1}{100}\right)} = 0.026;$$

$$z = \frac{\bar{p}_1 - \bar{p}_2}{s_{\bar{p}_1-\bar{p}_2}} = \frac{0.05 - 0.04}{0.026} = 0.385$$

Since z_{cal} (= 0.385) is less than its critical value $z_\alpha = 1.645$ at $\alpha = 0.05$, the H_0 is accepted.

10.20 Let $H_0 : p_1 \le p_2$ and $H_1 : p_1 > p_2$ (One-tailed test)

Given $n_1 = 500$, $\bar{p}_1 = 12/500 = 0.024$, $n_2 = 800$,

$\bar{p}_2 = 12/800 = 0.015$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} = \frac{12+12}{500+800} = 0.018;$$

$$q = 1 - p = 0.982$$

$$s_{\bar{p}_1-\bar{p}_2} = \sqrt{\bar{p}\,\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$= \sqrt{0.018 \times 0.912\left(\frac{1}{500} + \frac{1}{800}\right)} = 0.0076$$

$$z = \frac{\bar{p}_1 - \bar{p}_2}{s_{\bar{p}_1-\bar{p}_2}} = \frac{0.024 - 0.015}{0.0076} = 1.184$$

Since $z_{cal} = 1.184$ is less than its critical value $z_\alpha = 1.645$ at $\alpha = 0.05$, the H_0 is accepted. Production in second factory is better than in the first factory.

10.21 Let $H_0 : p_1 = p_2$ and $H_1 : p_1 \ne p_2$ (Two-tailed test)

Given $n = 480 + 520 = 1000$, $p = q = 0.5$.

$$z = \frac{\bar{p}_1 - \bar{p}_2}{\sigma_{\bar{p}_1-\bar{p}_2}} = \frac{\bar{p}_1 - \bar{p}_2}{\sqrt{npq}} = \frac{520 - 480}{\sqrt{1000\,(0.5)(0.5)}}$$

$$= \frac{40}{15.81} = 2.53$$

Since, $z_{cal} = 2.53$ is greater than its critical value $z_{\alpha/2} = 1.96$ at $\alpha/2 = 0.025$, the H_0 is rejected.

10.22 Let $H_0 : p = 4$ per cent and $H_1 : p \ne 4$ per cent (Two-tailed test)

Given $n = 600$, $\bar{p} = 36/600 = 0.06$

Confidence limits: $\bar{p} \pm z_\alpha \sqrt{\frac{pq}{n}}$

$$= 0.06 \pm 1.96\sqrt{(0.04 \times 0.96)/600}$$

$$= 0.06 \pm 0.016; \quad 0.44 \le p \le 0.076$$

Since probability 0.04 of the claim does not fall into the confidence limit, the claim is rejected.

10.10 HYPOTHESIS TESTING FOR POPULATION MEAN WITH SMALL SAMPLES

When the sample size is small (i.e., less than 30), the central limit theorem does not assure us to assume that the sampling distribution of a statistic such as mean $\bar{x}$, proportion $\bar{p}$, is normal. Consequently when testing a hypothesis with small samples, we must assume that the samples come from a normally or approximately normally distributed population. Under these conditions, the sampling distribution of sample statistic such as $\bar{x}$ and $\bar{p}$ is normal but the critical values of $\bar{x}$ or $\bar{p}$ depend on whether or not the population standard deviation σ is known. When the value of the population standard deviation σ is not known, its value is estimated by computing the standard deviation of sample s and the standard error of the mean is calculated by using the formula, $\sigma_{\bar{x}} = s/\sqrt{n}$. When we do this, the resulting sampling distribution may not be normal even if sampling is done from a normally distributed population. In all such cases the sampling distribution turns out to be the *Student's t-distribution.*

Sir William Gosset of Ireland in early 1900, under his pen name 'Student', developed a method for hypothesis testing popularly known as the '**t-test**'. It is said that Gosset was employed by Guinness Breway in Dublin, Ireland which did not permit him to publish his research findings under his own name, so he published his research findings in 1905 under the pen name 'Student'.

***t*-test:** A hypothesis test for comparing two independent population means using the means of two small samples.

10.10.1 Properties of Student's *t*-Distribution

If small samples of size n (≤ 30) are drawn from normal population with mean μ and for each sample we compute the sample statistic of interest, then probability density function of the t-distribution with degrees of freedom ν (a Greek letter *nu*) is given by

$$y = \frac{y_0}{\left(1+\frac{t^2}{n-1}\right)^{\frac{n}{2}}} = \frac{y_0}{\left(1+\frac{t^2}{\nu}\right)^{\frac{(\nu+1)}{2}}}\ ;\ -\infty \le t \le \infty$$

where y_0 is a constant depending on sample size n such that the total area under the curve is unity and $\nu = n - 1$.

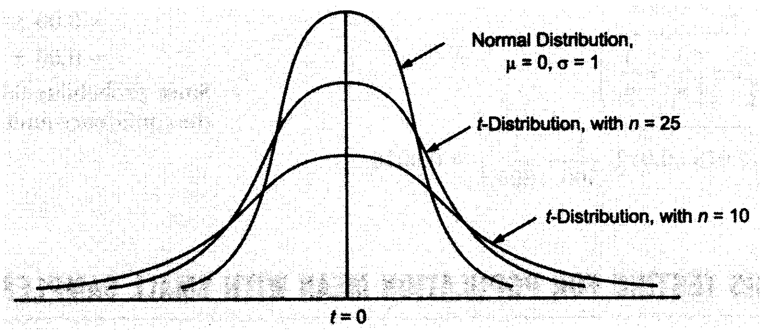
(i) As t appears in even power in probability density function, the t-distribution is symmetrical about the line $t = 0$ like the normal distribution.

(ii) The shape of the t-distribution depends on the sample size n. As n increases, the variability of t decreases. In other words, for large values of degrees of freedom, the t-distribution tends to a standard normal distribution. This implies that for different degrees of freedom, the shape of the t-distribution also differs, as shown in Fig. 10.9. Eventually when n is infinitely large, the t and z distributions are identical.

(iii) The t-distribution is less peaked than normal distribution at the centre and higher in the tails.

(iv) The t-distribution has greater dispersion than standard normal distribution with heavier tails, i.e. the t-curve does not approach x-axis as quickly as z does. This is because the t-statistic involves two random variables $\bar{x}$ and s, where as z-statistic involves only the sample mean $\bar{x}$. The variance of t-distribution is defined only when $\nu \ge 3$ and is given by var $(t) = \nu/(\nu - 2)$.

Figure 10.9
Comparison of t-Distribution with Standard Normal Distribution



(v) The value of y attains it maximum value at $t = 0$ so that the mode coincides with the mean. The limiting value of t-distribution when $\nu \to \infty$ is given by $y = y_0\, e^{-t^2/2}$. It follows that t is normally distributed for a large sample size,

(vi) The degrees of freedom refers to the number of independent squared deviations in s^2 that are available for estimating σ^2.

Uses of *t*-Distribution

There are various uses of t-distribution. A few of them are as follows:

(i) Hypothesis testing for the population mean.

(ii) Hypothesis testing for the difference between two populations means with independent samples.

(iii) Hypothesis testing for the difference between two populations means with dependent samples.

(iv) Hypothesis testing for an observed coefficient of correlation including partial and rank correlations.

(v) Hypothesis testing for an observed regression coefficient.

10.10.2 Hypothesis Testing for Single Population Mean

The test statistic for determining the difference between the sample mean $\bar{x}$ and population mean μ is given by

$$t = \frac{\bar{x} - \mu}{s_{\bar{x}}} = \frac{\bar{x} - \mu}{s/\sqrt{n}}\ ;\quad s = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n-1}}$$

where s is an unbiased estimation of unknown population standard deviation σ. This test statistic has a t-distribution with $n - 1$ degrees of freedom.

Confidence Interval The confidence interval estimate of the population mean μ when unknown population standard deviation σ is estimated by sample standard deviation s, is given by:

- Two-tailed test : $\bar{x} \pm t_{\alpha} \frac{s}{\sqrt{n}}$; α = level of significance
- One-tailed test : $\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$; α = level of significance

where t-test statistic value is based on a t-distribution with $n - 1$ degrees of freedom and $1 - \alpha$ is the confidence coefficient.

- Null hypothesis, H_0: $\mu = \mu_0$
- Alternative hypothesis:

One-tailed test	Two-tailed test
$H_1: \mu > \mu_0$ or $\mu < \mu_0$	$H_1: \mu \neq \mu_0$

Decision Rule: Rejected H_0 at the given degrees of freedom $n-1$ and level of significance when

One-tailed test	Two-tailed test
• $t_{cal} > t_{\alpha}$ or $t_{cal} < -t_{\alpha}$ for $H_1: \mu < \mu_0$	$t_{cal} > t_{\alpha/2}$ or $t_{cal} < -t_{\alpha/2}$
• Reject H_0 when p-value $< \alpha$	

Example 10.19: The average breaking strength of steel rods is specified to be 18.5 thousand kg. For this a sample of 14 rods was tested. The mean and standard deviation obtained were 17.85 and 1.955, respectively. Test the significance of the deviation.

Solution: Let us take the null hypothesis that there is no significant deviation in the breaking strength of the rods, that is,

$$H_0 : \mu = 18.5 \quad \text{and} \quad H_1 : \mu \neq 18.5 \quad \text{(Two-tailed test)}$$

Given, $n = 14$, $\bar{x} = 17.85$, $s = 1.955$, $df = n - 1 = 13$, and $\alpha = 0.05$ level of significance. The critical value of t at $df = 13$ and $\alpha/2 = 0.025$ is $t_{\alpha/2} = 2.16$.

Using the z-test statistic,

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{17.85 - 18.5}{\frac{1.955}{\sqrt{14}}} = -\frac{0.65}{0.522} = -1.24$$

Since t_{cal} (= − 1.24) value is more than its critical value $t_{\alpha/2} = -2.16$ at $\alpha/2 = 0.025$ and $df = 13$, the null hypothesis H_0 is accepted. Hence we conclude that there is no significant deviation of sample mean from the population mean.

Example 10.20: An automobile tyre manufacturer claims that the average life of a particular grade of tyre is more than 20,000 km when used under normal conditions. A random sample of 16 tyres was tested and a mean and standard deviation of 22,000 km and 5000 km, respectively were computed. Assuming the life of the tyres in km to be approximately normally distributed, decide whether the manufacturer's claim is valid.

Solution: Let us take the null hypothesis that the manufacturer's claim is valid, that is,

$$H_0 : \mu \geq 20{,}000 \quad \text{and} \quad H_1 : \mu < 20{,}000 \text{ (Left-tailed test)}$$

Given, $n = 16$, $\bar{x} = 22{,}000$, $s = 5000$, $df = 15$ and $\alpha = 0.05$ level of significance. The critical value of t at $df = 15$ and $\alpha = 0.05$ is $t_{\alpha} = 1.753$. Using the z-test statistic,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{22{,}000 - 20{,}000}{5000/\sqrt{16}} = \frac{2000}{1250} = 1.60$$

Since t_{cal} (= 1.60) value is less than its critical value $t_\alpha = 1.753$, $\alpha = 0.05$ and $df = 15$ at the null hypothesis H_0 is accepted. Hence we conclude that the manufacturer's claim is valid.

Example 10.21: A fertilizer mixing machine is set to give 12 kg of nitrate for every 100 kg of fertilizer. Ten bages of 100 kg each are examined. The percentage of nitrate so obtained is: 11, 14, 13, 12, 13, 12, 13, 14, 11, and 12. Is there reason to believe that the machine is defective?

Solution: Let us take the null hypothesis that the machine produces 12 kg of nitrate for every 100 kg of fertilizer, and is not defective, that is,

$$H_0 : \mu = 12 \quad \text{and} \quad H_1 : \mu \neq 12 \text{ (Two-tailed test)}$$

Given $n = 10$, $df = 9$, and $\alpha = 0.05$, critical value $t_{\alpha/2} = 2.262$ at $df = 9$ and $\alpha/2 = 0.025$.

Assuming that the weight of nitrate in bags is normally distributed and its standard deviation is unknown. The sample mean $\bar{x}$ and standard deviation s values are calculated as shown in Table 10.4.

Table 10.4: Calculations of Sample Mean $\bar{x}$ and Standard Deviation s

Variable x	Deviation, $d = x - 12$	d^2
11	−1	1
14	2	4
13	1	1
12	0	0
13	1	1
12	0	0
13	1	1
14	2	4
11	−1	1
12	0	0
125	5	13

$$\bar{x} = \frac{\Sigma x}{n} = \frac{125}{10} = 12.5 \text{ and } s = \sqrt{\frac{\Sigma (x-\bar{x})^2}{n-1}} = \sqrt{\frac{\Sigma d^2}{n-1} - \frac{(\Sigma d)^2}{n(n-1)}}$$

$$= \sqrt{\frac{13}{9} - \frac{(5)^2}{10(9)}} = 1.08$$

Using the z-test statistic, we have

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{12.5 - 12}{1.08/\sqrt{10}} = \frac{0.50}{0.341} = 1.466$$

Since t_{cal} (= 1.466) value is less than its critical value $t_{\alpha/2} = 2.262$, at $\alpha/2 = 0.025$ and $df = 9$, the null hypothesis H_0 is accepted. Hence we conclude that the manufacturer's claim is valid, that is, the machine is not defective.

Example 10.22: A random sample of size 16 has the sample mean 53. The sum of the squares of deviation taken from the mean value is 150. Can this sample be regarded as taken from the population having 56 as its mean? Obtain 95 per cent and 99 per cent confidence limits of the sample mean.

Solution: Let us take the null hypothesis that the population mean is 56, i.e.

$$H_0 : \mu = 56 \quad \text{and} \quad H_1 : \mu \neq 56 \text{ (Two-tailed test)}$$

$$\text{Given, } n = 16, df = n - 1 = 15, \bar{x} = 53; s = \sqrt{\frac{\Sigma (x - \bar{x})^2}{(n-1)}} = \sqrt{\frac{150}{15}} = 3.162$$

- 95 per cent confidence limit

$$\bar{x} \pm t_{0.05} \frac{s}{\sqrt{n}} = 53 \pm 2.13 \frac{3.162}{\sqrt{16}} = 53 \pm 2.13\,(0.790) = 53 \pm 1.683$$

- 99 per cent confidence limit

$$\bar{x} \pm t_{0.01} \frac{s}{\sqrt{n}} = 53 \pm 2.95 \frac{3.162}{\sqrt{16}} = 53 \pm 2.33$$

10.10.3 Hypothesis Testing for Difference of Two Population Means (Independent Samples)

For comparing the mean values of two normally distributed populations, we draw independent random samples of sizes n_1 and n_2 from the two populations. If μ_1 and μ_2 are the mean values of two populations, then our aim is to estimate the value of the difference $\mu_1 - \mu_2$ between mean values of the two populations.

Since sample mean $\bar{x}_1$ and $\bar{x}_2$ are the best point estimators to draw inferences regarding μ_1 and μ_2 respectively, therefore the difference between the sample means of the two independent simple random samples, $\bar{x}_1 - \bar{x}_2$ is the best point estimator of the difference $\mu_1 - \mu_2$.

The sampling distribution of $\bar{x}_1 - \bar{x}_2$ has the following properties:

- Expected value : $E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu_1 - \mu_2$

This implies that the sample statistic $(\bar{x}_1 - \bar{x}_2)$ is an unbiased point estimator of $\mu_1 - \mu_2$.

- Variance : $\text{Var}\,(\bar{x}_1 - \bar{x}_2) = \text{Var}\,(\bar{x}_1) + \text{Var}\,(\bar{x}_2) = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}$

If the population standard deviations σ_1 and σ_2 are known, then the large sample interval estimation can also be used for the small sample case. But if these are unknown, then these are estimated by the sample standard deviations s_1 and s_2. It is needed if sampling distribution is not normal even if sampling is done from two normal populations. The logic for this is the same as that for a single population case. Thus t-distribution is used to develop a small sample interval estimate for $\mu_1 - \mu_2$.

Population Variances are Unknown But Equal

If population variances σ_1^2 and σ_2^2 are unknown but equal, that is, both populations have exactly the same shape and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then standard error of the difference in two sample means $\bar{x}_1 - \bar{x}_2$ can be written as:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

In such a case we need not estimate σ_1^2 and σ_2^2 separately and therefore data from two samples can be combined to get a pooled, single estimate of σ^2. If we use the sample estimate s^2 for the population varriance σ^2, then the pooled variance estimator of σ^2 is given by

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)\,s_2^2}{(n_1 - 1) + (n_2 - 1)} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)\,s_2^2}{n_1 + n_2 - 2}$$

This single variance estimator s^2 is a *weighted average* of the values of s_1^2 and s_2^2 in which weights are based on the degrees of freedom $n_1 - 1$ and $n_2 - 1$. Thus the point estimate of $\sigma_{\bar{x}_1 - \bar{x}_2}$ when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is given by

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Since $s_1 = \sqrt{\dfrac{\Sigma (x_1 - \bar{x}_1)^2}{(n_1 - 1)}}$ and $s_2 = \sqrt{\dfrac{\Sigma (x_2 - \bar{x}_2)^2}{(n_2 - 1)}}$, therefore the pooled variance s^2 can also be calculated as

$$s^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

Following the same logic as discussed earlier, the t-test statistic is defined as

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1 - \bar{x}_2}} = \frac{\bar{x}_1 - \bar{x}_2}{s} \sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

The sampling distribution of this t-statistic is approximated by the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Null hypothesis: $H_0 : \mu_1 - \mu_2 = d_0$	
Alternative hypothesis:	
One-tailed Test	**Two-tailed Test**
$H_1 : (\mu_1 - \mu_2) > d_0$ or $(\mu_1 - \mu_2) < d_0$	**$H_1 : \mu_1 - \mu_2 \neq d_0$**

Decision Rule Rejected H_0 at $df = n_1 + n_2 - 2$ and at specified level of significance α when

One-tailed test	Two-tailed test
$t_{cal} > t_\alpha$ or $t_{cal} < -t_\alpha$ for $H_1 : (\mu_1 - \mu_2) < d_0$	$t > t_{\alpha/2}$ or $t_{cal} < -t_{\alpha/2}$

Confidence Interval: The confidence interval estimate of the difference between populations means for small samples of size $n_1 < 30$ and/or $n_2 < 30$ with unknown σ_1 and σ_2 estimated by s_1 and s_2 is given by

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \, s_{\bar{x}_1 - \bar{x}_2}$$

where $t_{\alpha/2}$ is the critical value of t. The value of $t_{\alpha/2}$ depends on the t-distribution with $n_1 + n_2 - 2$ degrees of freedom and confidence coefficient $1 - \alpha$.

Population Variances are Unknown and Unequal

When two population variances are not equal, we may estimate the standard error $\sigma_{\bar{x}_1 - \bar{x}_2}$ of the statistic $(\bar{x}_1 - \bar{x}_2)$ by sample variances s_1^2 and s_2^2 in place of σ_1^2 and σ_2^2. Thus an estimate of standard error of $\bar{x}_1 - \bar{x}_2$ is given by

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

The sampling distribution of a t-test statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1 - \bar{x}_2}} = \frac{\bar{x}_1 - x_2}{s_{\bar{x}_1 - \bar{x}_2}}$$

is approximated by t-distribution with degrees of freedom given by

$$\text{Degrees of freedom } (df) = \frac{[s_1^2/n_1 + s_2^2/n_2]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

The number of degrees of freedom in this case is less than that obtained is Case 1 above.

Example 10.23: In a test given to two groups of students, the marks obtained are as follows:

First group :	18	20	36	50	49	36	34	49	41
Second group :	29	28	26	35	30	44	46		

Examine the significance of the difference between the arithmetic mean of the marks secured by the students of the above two groups.

[Madras Univ., MCom, 1997; MD Univ., MCom, 1998]

Solution: Let us take the null hypothesis that there is no significant difference in arithmetic mean of the marks secured by students of the two groups, that is,

$$H_0 : \mu_1 - \mu_2 = 0 \text{ or } \mu_1 = \mu_2 \quad \text{and} \quad H_1 : \mu_1 \neq \mu_2 \quad \text{(Two-tailed test)}$$

Since sample size in both the cases is small and sample variances are not known, apply t-test statistic to test the null hypothesis

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

Calculations of sample means $\bar{x}_1$, $\bar{x}_2$ and pooled sample standard deviations are shown in Table 10.5.

Table 10.5: Calculation for $\bar{x}_1$, $\bar{x}_2$ and s

First Group x_1	$x_1 - \bar{x}_1 = x_1 - 37$	$(x_1 - \bar{x}_1)^2$	Second Group x_2	$x_2 - \bar{x}_2 = x_2 - 34$	$(x_2 - \bar{x}_2)^2$
18	−19	361	29	−5	25
20	−17	389	28	−6	36
36	−1	1	26	−8	64
50	13	169	35	1	1
49	12	144	30	−4	16
36	−1	1	44	10	100
34	−3	9	46	12	144
49	12	144			
41	4	16			
333	0	1,234	238	0	386

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{333}{9} = 37 \quad \text{and} \quad \bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{238}{7} = 34$$

$$s = \sqrt{\frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}} = \sqrt{\frac{1234 + 386}{9 + 7 - 2}} = 10.76$$

Substituting values in the t-test statistic, we get

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}} = \frac{37 - 34}{10.76}\sqrt{\frac{9 \times 7}{9 + 7}} = \frac{3}{10.46} \times 1.984 = 0.551$$

Degrees of freedom, $df = n_1 + n_2 - 2 = 9 + 7 - 2 = 14$

Since at $\alpha = 0.05$ and $df = 14$, the calculated value t_{cal} (= 0.551) is less than its critical value $t_{\alpha/2} = 2.14$, the null hypothesis H_0 is accepted. Hence we conclude that the mean marks obtained by the students of two groups do not differ significantly.

Example 10.24: The mean life of a sample of 10 electric light bulbs was found to be 1456 hours with standard deviation of 423 hours. A second sample of 17 bulbs chosen from a different batch showed a mean life of 1280 hours with standard deviation of 398 hours. Is there a significant difference between the means of the two batches.

[Delhi Univ, MCom, 1997]

Solution: Let us take the null hypothesis that there is no significant difference between the mean life of electric bulbs of two batches, that is,

$$H_0 : \mu_1 = \mu_2 \quad \text{and} \quad H_1 : \mu_1 \neq \mu_2 \quad \text{(Two-tailed test)}$$

Given, $n_1 = 10$, $\bar{x}_1 = 1456, s_1 = 423; n_2 = 17, \bar{x}_2 = 1280, s_2 = 398$ and $\alpha = 0.05$. Thus,

$$\text{Pooled standard deviation, } s = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{9(423)^2 + 16(398)^2}{10+17-2}}$$

$$= \sqrt{\frac{16{,}10{,}361 + 25{,}34{,}464}{25}} = \sqrt{1{,}65{,}793} = 407.18$$

Applying the t-test, we have

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}} = \frac{1456-1280}{407.18}\sqrt{\frac{10\times 17}{10+17}}$$

$$= \frac{176}{407.18} \times 2.51 = 1.085$$

Since the calculated value $t_{cal} = 1.085$ is less than its critical value $t_{\alpha/2} = 2.06$ at $df = 25$ and $\alpha = 0.05$ level of significance, the null hypothesis is accepted. Hence we conclude that the mean life of electric bulbs of two batches does not differ significantly.

Example 10.25: The manager of a courier service believes that packets delivered at the end of the month are heavier than those delivered early in the month. As an experiment, he weighed a random sample of 20 packets at the beginning of the month. He found that the mean weight was 5.25 kgs with a standard deviation of 1.20 kgs. Ten packets randomly selected at the end of the month had a mean weight of 4.96 kgs and a standard deviation of 1.15 kgs. At the 0.05 significance level, can it be concluded that the packets delivered at the end of the month weigh more?

Solution: Let us take the null hypothesis that the mean weight of packets delivered at the end of the month is more than the mean weight of packets delivered at the beginning of the month, that is

$$H_0 : \mu_E \geq \mu_B \quad \text{and} \quad H_1 : \mu_E < \mu_B$$

Given $n_1 = 20$, $\bar{x}_1 = 5.25, s_1 = 1.20$ and $n_2 = 10$, $\bar{x}_2 = 4.96, s_2 = 1.15$. Thus

$$\text{Pooled standard deviation, } s = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{19\times(5.25)^2 + 9(4.96)^2}{20+10-2}}$$

$$= \sqrt{\frac{19\times 27.56 + 9\times 24.60}{28}} = \sqrt{26.60} = 5.16$$

Applying the t-test, we have

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}} = \frac{5.25-4.96}{5.16}\sqrt{\frac{20\times 10}{20+10}}$$

$$= \frac{0.29}{5.16}\sqrt{\frac{200}{30}} = 0.056\times 2.58 = 0.145$$

Since at $\alpha = 0.01$ and $df = 28$, the calculated value t_{cal} (= 0.145) is less than it's critical value $z_\alpha = 1.701$, the null hypothesis is accepted. Hence, packets delivered at the end of the month weigh more on an average.

10.10.4 Hypothesis Testing for Difference of Two Population Means (Dependent Samples)

When two samples of the same size are paired so that each observation in one sample is associated with any particular observation in the second sample, the sampling procedure to collect the data and then test the hypothesis is called *matched samples.* In such a case the 'difference' between each pair of data is first calculated and then the these differences are treated as a single set of data in order to consider whether there has been any significant change or whether the differences could have occurred by chance.

The matched sampling plan often leads to a smaller sampling error than the independent sampling plan because in matched samples variation as a source of sampling error is eliminated.

Let μ_d be the mean of the difference values for the population. Then this mean value μ_d is compared to zero or some hypothesized value using the *t*-test for a single sample. The *t*-test statistic is used because the standard deviation of the population of differences is unknown, and thus the statistical inference about μ_d based on the average of the sample differences $\bar{d}$ would involve the *t*-distribution rather than the standard normal distribution. The *t*-test, also called *paired t-test*, becomes

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

where n = number of paired observations

$df = n - 1$, degrees of freedom

$\bar{d}$ = mean of the difference between paired (or related) observeations

n = number of pairs of differences

s_d = sample standard deviation of the distribution of the difference between the paired (or related observations)

$$= \sqrt{\frac{\Sigma (d - \bar{d})^2}{n-1}} = \sqrt{\frac{\Sigma d^2}{n-1} - \frac{(\Sigma d)^2}{n(n-1)}}$$

The null and alternative hypotheses are stated as:

$H_0 : \mu_d = 0$ or c (Any hypothesized value)

$H_1 : \mu_d > 0$ or $(\mu_d < 0)$ (One-tailed Test)

$\mu_d \neq 0$ (Two-tailed Test)

Decision rule: If the calculated value t_{cal} is less than its critical value, t_d at a specified level of significance and known degrees of freedom, then null hypothesis H_0 is accepted. Otherwise H_0 is rejected.

Confidence interval: The confidence interval estimate of the difference between two population means is given by

$$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$$

where $t_{\alpha/2}$ = critical value of *t*-test statistic at $n - 1$ degrees of freedom and α level of significance.

If the claimed value of null hypothesis H_0 lies within the confidence interval, then H_0 is accepted, otherwise rejected.

Example 10.26: The HRD manager wishes to see if there has bean any change in the ability of trainees after a specific training programme. The trainees take an aptitude test before the start of the programme and an equivalent one after they have completed it. The scores recorded are given below. Has any change taken place at 5 per cent significance level?

Trainee	A	B	C	D	E	F	G	H	I
Score before training	75	70	46	68	68	43	55	68	77
Score after training	70	77	57	60	79	64	55	77	76

Solution: Let us take the null hypothesis that there is no change taken place after the training, that is,

$$H_0 : \mu_d = 0 \quad \text{and} \quad H_1 : \mu_d \neq 0 \quad \text{(Two-tailed test)}$$

The 'changes' are computed as shown in Table 10.6 and then a t-test is carried out on these differences as shown below.

Table 10.6: Calculations of 'Changes'

Trainee	Before Training	After Training	Difference in Scores, d	d^2
A	75	70	5	25
B	70	77	7	49
C	46	57	−11	121
D	68	60	8	64
E	68	79	−11	121
F	43	64	−21	441
G	55	55	0	0
H	68	77	−9	81
I	77	76	1	1
			−45	903

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-45}{9} = -5 \text{ and}$$

$$s_d = \sqrt{\frac{\Sigma d^2}{n-1} - \frac{(\Sigma d)^2}{n(n-1)}} = \sqrt{\frac{903}{8} - \frac{(-45)^2}{9 \times 8}} = \sqrt{112.87 - 28.13} = 9.21$$

Applying the t-test statistic, we have

$$t = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} = \frac{-5-0}{9.21/\sqrt{9}} = -\frac{5}{3.07} = -1.63$$

Since the calculated value $t_{cal} = -1.63$ is more than its critical value, $t_{\alpha/2} = -2.31$, at $df = 8$ and $\alpha/2 = 0.025$ the null hypothesis is accepted. Hence, we conclude that there is no change in the ability of trainees after the training.

Example 10.27: 12 students were given intensive coaching and 5 tests were conducted in a month. The scores of tests 1 and 5 are given below.

No. of students :	1	2	3	4	5	6	7	8	9	10	11	12
Marks in 1st test :	50	42	51	26	35	42	60	41	70	55	62	38
Marks in 5th test :	62	40	61	35	30	52	68	51	84	63	72	50

Do the data indicate any improvement in the scores obtained in tests 1 and 5

[*Punjab Univ., MCom, 1999*]

Solution: Let us take the hypothesis that there is no improvement in the scores obtained in the first and fifth tests, that is,

$$H_0 : \mu_d = 0 \quad \text{and} \quad H_1 : \mu_d \neq 0 \quad \text{(Two-tailed test)}$$

The 'changes' are calculated as shown in Table 10.7 and then t-test is carried out on these differences as shown below: